ON THE BOUNDEDNESS IN $H^{1/4}$ OF THE MAXIMAL SQUARE FUNCTION ASSOCIATED WITH THE SCHRÖDINGER EQUATION

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Abstract. A long standing conjecture for the linear Schrödinger equation states that 1/4 of derivative in L^2, in the sense of Sobolev spaces, suffices in any dimension for the solution to that equation to converge almost everywhere to the initial datum as the time goes to 0. This is only known to be true in dimension 1 by work of Carleson. In this paper we show that the conjecture is true on spherical averages. To be more precise, we prove the L^2 boundedness of the associated maximal square function on the Sobolev class $H^{1/4}(\mathbb{R}^n)$ in any dimension n.

1. Introduction

For $\alpha \in \mathbb{R}$, we denote by $H^\alpha(\mathbb{R}^n)$ the Sobolev space

$$H^\alpha(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^\alpha} = \left(\int \left| \widehat{f}(\xi) \right|^2 (1+|\xi|^2)^\alpha \, \mathrm{d}\xi \right)^{1/2} < \infty \right\}.$$

We will also consider the homogeneous Sobolev space $\dot{H}^\alpha(\mathbb{R}^n)$ defined by

$$\dot{H}^\alpha(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{H}^\alpha} = \left(\int \left| \widehat{f}(\xi) \right|^2 |\xi|^{2\alpha} \, \mathrm{d}\xi \right)^{1/2} < \infty \right\}.$$

Let f be in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, and define

$$S_t f(x) = u(x,\, t) = \int_{\mathbb{R}^n} \widehat{f}(\xi) \mathrm{e}^{-2\pi \mathrm{i} |\xi|^2 t} \mathrm{e}^{2\pi \mathrm{i} \xi \cdot x} \, \mathrm{d}\xi.$$

Then u is the solution to the linear Schrödinger equation with initial datum f, that is,

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \frac{\mathrm{i}}{2\pi} \Delta_x u(x,t) & \text{in } \mathbb{R}^{n+1}_+ \\ u(x,0) = f(x) & \text{in } \mathbb{R}^n. \end{cases}$$

There is a fundamental question in this setting and is that of determining the minimal smoothness on the initial value function f, needed for the almost everywhere convergence

$$\lim_{t \to 0^+} u(x,\, t) = f(x), \qquad \text{a.e.} \tag{1.1}$$

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This smoothness is measured in terms of the Sobolev space H^α which the function f belongs to. In 1979, Carleson proved in [4] that the a.e. convergence (1.1) holds for any $f \in \dot{H}^{1/4}$ in dimension $n = 1$. Dahlberg and Kenig [6] extended this result to functions in $\dot{H}^{n/4}(\mathbb{R}^n)$ for any n and showed that there are counterexamples if the regularity is less than 1/4. It is conjectured that $\alpha = 1/4$ suffices for this problem in any dimension n. Sjölin and Vega proved independently in [12], [16] that α greater than 1/2 implies the convergence (1.1) in any dimension n (previous results, for $\alpha > 1$, were obtained in [3], [5]), while Prestini [11] proved the conjecture for radial functions. The case $n = 2$ has been intensively studied during the last years and is the only one (apart from $n = 1$) where there are positive results for (1.1) with smoothness $\alpha < 1/2$ (see [10], [14], [15], and the references there).

As usual, problems related to the a.e. convergence are intimately connected to the boundedness of the associated maximal function. In our case, this maximal function is given by $S^* f(x) = \sup_{t>0} |S_t f(x)|$, $x \in \mathbb{R}^n$. For example, the a.e. convergence (1.1) for all functions $f \in H^\alpha$ follows from the a priori maximal estimate

$$\left(\int_{|x|\leq 1} |S^* f(x)|^p \, \mathrm{d}x \right)^{1/p} \leq C \|f\|_{H^\alpha}, \qquad f \in \mathcal{S}(\mathbb{R}^n). \tag{1.2}$$

In fact, all the known cases about convergence mentioned above are obtained via this maximal inequality for different values of $p \in [1, 2]$.

In this paper, we investigate whether inequality (1.2) holds if we replace S^* by a spherical average operator; namely we look at the maximal square function

$$Q^* f(x) = \sup_{t>0} \left(\frac{1}{\sigma(S^{n-1})} \int_{S^{n-1}} |S_t f(|x|\omega)|^2 \, \mathrm{d}\sigma(\omega) \right)^{1/2}.$$

Clearly, one has the inequality $\int_{|x|\leq 1} |Q^* f(x)|^2 \, \mathrm{d}x \leq \int_{|x|\leq 1} |S^* f(x)|^2 \, \mathrm{d}x$, and therefore the boundedness of S^* would imply a corresponding inequality for Q^*. The known counterexamples show that the smoothness condition $\alpha \geq 1/4$ is still necessary for the boundedness of this operator. The main result of this paper says that $\alpha = 1/4$ is also sufficient for the boundedness of Q^*.

Theorem 1.1. *The operator Q^* is bounded from $\dot{H}^{1/4}(\mathbb{R}^n)$ into $L^2(\{|x| \leq 1\})$ in any dimension n; in fact, there is a positive constant C, independent of the dimension, such that*

$$\left(\int_{|x|\leq 1} |Q^* f(x)|^2 \, \mathrm{d}x \right)^{1/2} \leq C \|f\|_{\dot{H}^{1/4}}, \qquad \forall f \in \mathcal{S}(\mathbb{R}^n). \tag{1.3}$$

In particular, (1.3) gives us that 1/4 of smoothness suffices for the a.e. convergence with respect to quadratic spherical means. The precise statement is contained in the following corollary.

Corollary 1.2. *If $f \in H^{1/4}(\mathbb{R}^n)$, then, for every $x_0 \in \mathbb{R}^n$ we have*

$$\lim_{t\to 0^+} \int_{S^{n-1}} |S_t f(x_0 + r\omega) - f(x_0 + r\omega)|^2 \, \mathrm{d}\sigma(\omega) = 0, \quad \textit{a.e. } r.$$

Proof. The proof is standard. By translation invariance, we may assume without loss of generality that $x_0 = 0$. It is easy to see that, if $g \in \mathcal{S}(\mathbb{R}^n)$, then $S_t g \to g$ as $t \to 0^+$, uniformly in $\mathbb{R}^n$. Given $f \in H^{1/4}(\mathbb{R}^n)$ we take a sequence $\{g_k\}_{k=1}^\infty \subseteq \mathcal{S}(\mathbb{R}^n)$

such that $g_k \to f$ in $H^{1/4}(\mathbb{R}^n)$. Denote by μ the Borel measure $\mathrm{d}\mu(r) = r^{n-1}\,\mathrm{d}r$. Let $\lambda > 0$, $B^n = \{x \in \mathbb{R}^n \colon |x| \le 1\}$ and define

$$A_\lambda = \left\{0 < r < 1 \colon \limsup_{t\to 0^+} \int_{S^{n-1}} |S_t f(r\omega) - f(r\omega)|^2 \,\mathrm{d}\sigma(\omega) > \lambda\right\}.$$

Then, for any positive integer k

$$\begin{aligned}\mu(A_\lambda) \quad &\le \quad \mu\left(\left\{0 < r < 1 \colon \limsup_{t\to 0^+} \int_{S^{n-1}} |S_t(f-g_k)(r\omega)|^2 \,\mathrm{d}\sigma(\omega) > \frac{\lambda}{2}\right\}\right) \\ &+ \quad \mu\left(\left\{0 < r < 1 \colon \int_{S^{n-1}} |g_k(r\omega) - f(r\omega)|^2 \,\mathrm{d}\sigma(\omega) > \frac{\lambda}{2}\right\}\right).\end{aligned}$$

Now, Chebyshev's inequality and Theorem 1.1 imply that

$$\begin{aligned}\mu(A_\lambda) \quad &\le \quad \frac{C}{\lambda}\int_{B^n} \sup_{t>0} \int_{S^{n-1}} |S_t(f-g_k)(|x|\omega)|^2 \,\mathrm{d}\sigma(\omega)\,\mathrm{d}x \\ &+ \quad \frac{C}{\lambda}\|f-g_k\|_2^2 < \frac{C}{\lambda}\|f-g_k\|_{H^{1/4}}^2, \quad \forall\ k,\end{aligned}$$

and, therefore, $\mu(A_\lambda) = 0$. □

Before we proceed with the proof of Theorem 1.1, let us first make a reformulation of our problem and some additional comments. Observe that if $\{\mathcal{Y}_k\}$ is an orthonormal basis of spherical harmonics in $L^2(S^{n-1})$, and $\hat{f}(\xi) \sim \sum_k f_k(|\xi|)\mathcal{Y}_k(\xi/|\xi|)$ denotes the corresponding expansion of $\hat{f}$ with respect to this basis, then

$$Q^* f(x) = \sup_{t>0} \left(\frac{2\pi}{\sigma\left(S^{n-1}\right)} \sum_k \frac{1}{|x|^{n-1}} \left|Q^t_{\nu(k)}\left(f_k(s)s^{(n-1)/2}\right)(|x|)\right|^2\right)^{1/2},$$

where

$$Q^t_\nu g(r) = \int_0^\infty \mathrm{e}^{-2\pi\mathrm{i}ts^2}\widetilde{J}_\nu(2\pi rs)g(s)\,\mathrm{d}s$$

and $\nu(k) = (n-2)/2 + \mathrm{degree}(\mathcal{Y}_k)$. Here, J_ν denotes the Bessel function of order ν and $\widetilde{J}_\nu(t) = \sqrt{t}J_\nu(t)$ for $t \ge 0$. Using that the norm in $\dot{H}^{1/4}$ of f with respect to the above expansion is given by $\|f\|_{\dot{H}^{1/4}} = \sum_k \int_0^\infty |f_k(r)|^2 r^{1/2} r^{n-1}\,\mathrm{d}r$, and "cancelling out the $\sum$ signs", the inequality

$$\int_{|x|\le 1} |Q^* f(x)|^2\,\mathrm{d}x \le C\|f\|_{\dot{H}^{1/4}}^2$$

is equivalent to the estimate

$$\int_0^1 \sup_{t>0} |Q^t_\nu g(r)|^2\,\mathrm{d}r \le C\int |g(r)|^2 r^{1/2}\,\mathrm{d}r,$$

uniformly in the index ν too.

We can now follow Carleson's approach (see [4], [6]). First we linearize our maximal operator, by making t into a function of r, $t(r)$. Next we may assume that g is supported on a fixed interval I (as long as the final constant C is independent of I). "Moving" the smoothness to the other side (that is, redefining $g(r)r^{1/4}$ as g again), we consider instead the linear operator

$$T_\nu g(r) = \int_I \frac{\mathrm{e}^{\mathrm{i}s^2 t(r)}\widetilde{J}_\nu(rs)}{s^{1/4}} g(s)\,\mathrm{d}s.$$

Then what we have to show is

$$\int_0^1 |T_\nu g(r)|^2 \,\mathrm{d}r \le C \int_I |g(s)|^2 \,\mathrm{d}s, \tag{1.4}$$

with C independent of $g \in L^2(I)$, of the interval I, of the measurable function $t(r)$ and of $\nu \in \mathbb{N}/2$.

We want to point out that Theorem 1.1 gives as a consequence the boundedness of the maximal Schrödinger operator S^* on radial functions in $\mathbb{R}^n$, with constant independent of n. A close look at the above arguments will convince us that both, Theorem 1.1 and this dimension-free estimate are, in fact, equivalent.

Let us bring here a related result obtained by the authors. In [8] it was proved that the uniform estimate

$$\int_I \mathrm{e}^{\mathrm{i}as^2} J_\nu(s) \frac{\mathrm{d}s}{s^\beta} = O(1),$$

independent of $\nu \in \mathbb{N}/2$, the interval I and $a \in \mathbb{R}$, holds (for $\beta < 1$) if and only if $\beta \ge 1/6$. This expression appears in a natural way as the leading term (using the product formula for Bessel functions) of the expansion of the kernel associated to $T_\nu T_\nu^*$ but replacing the "smoothness" $1/4$ by the generic smoothness α with $2\alpha - 1/2 = \beta$. This could be interpreted as an indication that the uniform estimate of the operators Q_ν by this method would only be possible on the class $\dot{H}^{1/3}$ ($\alpha = 1/3$ corresponds to the case $\beta = 1/6$). Our theorem here shows that an additional cancellation of the rest of terms in the expansion of the kernel is possible so that, as Theorem 1.1 says, the result holds indeed on $\dot{H}^{1/4}$.

Continuing with the reduction of our problem, let us point out that by using a TT^* argument and the well known expansion

$$\widetilde{J}_\nu(r) = \sqrt{\frac{2}{\pi}} \cos\left(r - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O_\nu\left(\frac{1}{r}\right) \qquad \text{as } r \to \infty,$$

it is not difficult to obtain (1.4) but with a constant which would depend on ν (see also [11]). Thus we only need to check that the constant C is uniformly bounded as ν tends to infinity.

The following lemma, due to J. A. Barceló ([1], [2]), describes the oscillation and the asymptotics of the Bessel function for large values, with the precise dependence of the remainder term with respect to the order of the function.

Lemma 1.3. *There is a universal constant $C > 0$ such that for all $\nu > 1/2$ and for all $r > \nu + \nu^{1/3}$ we have*

$$J_\nu(r) = \sqrt{\frac{2}{\pi}} \frac{\cos\theta(r)}{(r^2 - \nu^2)^{1/4}} + h_\nu(r),$$

where

$$\theta(r) = (r^2 - \nu^2)^{1/2} - \nu \arccos\frac{\nu}{r} - \frac{\pi}{4},$$

and

$$|h_\nu(r)| \le \begin{cases} C\left(\frac{\nu^2}{(r^2-\nu^2)^{7/4}} + \frac{1}{r}\right) & \text{if } \nu + \nu^{1/3} \le r \le 2\nu \\ \frac{C}{r} & \text{if } r \ge 2\nu. \end{cases}$$

In order to simplify the notation, let us define for $r > \nu + \nu^{1/3}$ the functions

$$\begin{aligned} J_\nu^B(r) &= \sqrt{\frac{2}{\pi}} \frac{\cos\theta(r)}{(r^2-\nu^2)^{1/4}}, \\ \widetilde{J}_\nu^B(r) &= \sqrt{r} J_\nu^B(r), \\ \widetilde{h}_\nu(r) &= \sqrt{r} h_\nu(r). \end{aligned}$$

Thus, we can write T_ν as the sum of the following operators

$$\begin{aligned} T_\nu^1 g(r) &= \int_I \mathrm{e}^{\mathrm{i}t(r)s^2} \widetilde{J}_\nu(rs)\chi_{[0,\,\nu]}(rs) s^{-1/4} g(s)\,\mathrm{d}s, \\ T_\nu^2 g(r) &= \int_I \mathrm{e}^{\mathrm{i}t(r)s^2} \widetilde{J}_\nu(rs)\chi_{[\nu,\,\nu+\nu^{2/3}]}(rs) s^{-1/4} g(s)\,\mathrm{d}s, \\ T_\nu^3 g(r) &= \int_I \mathrm{e}^{\mathrm{i}t(r)s^2} \tilde{h}_\nu(rs)\chi_{[\nu+\nu^{2/3},\,2\nu]}(rs) s^{-1/4} g(s)\,\mathrm{d}s, \\ T_\nu^4 g(r) &= \int_I \mathrm{e}^{\mathrm{i}t(r)s^2} \widetilde{J}_\nu^B(rs)\chi_{[\nu+\nu^{2/3},\,2\nu]}(rs) s^{-1/4} g(s)\,\mathrm{d}s, \\ T_\nu^5 g(r) &= \int_I \mathrm{e}^{\mathrm{i}t(r)s^2} \tilde{h}_\nu(rs)\chi_{[2\nu,\,\infty)}(rs) s^{-1/4} g(s)\,\mathrm{d}s, \\ T_\nu^6 g(r) &= \int_I \mathrm{e}^{\mathrm{i}t(r)s^2} \widetilde{J}_\nu^B(rs)\chi_{[2\nu,\,\infty)}(rs) s^{-1/4} g(s)\,\mathrm{d}s. \end{aligned}$$

The desired boundedness will now follow from the boundedness of the above operators. This will be proved in sections 2 through 6, but first we would like to recall Van der Corput's lemma.

Lemma 1.4 (Van der Corput). *Let ϕ be a smooth real valued function defined on an interval $[a,b]$ and ψ a smooth positive decreasing function defined on the same interval. Suppose that ϕ' is monotonic in $[a,b]$ and that $|\phi'(s)| \geq \lambda$ for all $s \in [a,b]$. Then there is a universal constant $C > 0$ such that*

$$\left| \int_a^b \mathrm{e}^{\mathrm{i}\phi(s)} \psi(s)\,\mathrm{d}s \right| \leq C \frac{\psi(a)}{\lambda}.$$

A proof of this can be found in [13].

2. Boundedness of T_ν^1

We need the following version of Schur's lemma.

Lemma 2.1. *Given two σ-finite measure spaces $(X,\,\mu)$, $(Y,\,\nu)$ and a $\mu\otimes\nu$-measurable function k on $X \times Y$, suppose that there exists a positive constant C such that*

$$\sup_{u\in X} \int_Y \int_X |k(x,\,y)k(u,\,y)|\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y) < C.$$

Then, if $f \in L^2(X,\,\mu)$, the integral

$$Kf(y) = \int_X k(x,\,y) f(x)\,\mathrm{d}\mu(x)$$

converges absolutely for a.e. $y \in Y$, the function Kf thus defined is in $L^2(Y,\,\nu)$ and

$$\|Kf\|_2^2 \leq C\|f\|_2^2.$$

The next proposition discusses the boundedness of the operator T^1_ν.

Proposition 2.2. *There is a positive constant C such that for all $\nu \geq 1$, for all intervals I, for all measurable functions $t(r)$ and for all $g \in L^2(I)$,*

$$\|T^1_\nu g\|_{L^2(0,\,1)} \leq C\|g\|_{L^2(I)}.$$

proof. The kernel of the operator T^1_ν is

$$k(s,\, r) = \mathrm{e}^{\mathrm{i}t(r)s^2} \widetilde{J}_\nu(sr)\chi_{[0,\,\nu]}(sr)s^{-1/4},$$

so that $|k(s,\, r)| = |\widetilde{J}_\nu(sr)|\chi_{[0,\,\nu]}(sr)s^{-1/4}$. By Lemma 2.1,

$$\begin{aligned}
\|T^1_\nu\|_2^2 &\leq \sup_{u\in I} \int_0^1 \widetilde{J}_\nu(uy)\chi_{[0,\,\nu]}(uy)u^{-1/4} \int_I \widetilde{J}_\nu(sy)\chi_{[0,\,\nu]}(sy)s^{-1/4}\,\mathrm{d}s\,\mathrm{d}y \\
&= \sup_{u\in I} \int_0^1 \widetilde{J}_\nu(uy)\chi_{[0,\,\nu]}(uy)u^{-1/4}y^{-3/4} \int_{I'\cap[0,\,\nu]} \widetilde{J}_\nu(t)t^{-1/4}\,\mathrm{d}t\,\mathrm{d}y.
\end{aligned}$$

Since, by the well-known estimates for J_ν in the interval $[0,\, \nu/2]$ (see [17]) and Stirling's formula,

$$\begin{aligned}
\int_0^{\nu/2} \frac{J_\nu(t)}{t^\gamma}\,\mathrm{d}t &\leq \int_0^{\nu/2} \frac{t^{\nu-\gamma}}{2^\nu\Gamma(\nu+1)}\,\mathrm{d}t = \frac{\nu^{\nu+1-\gamma}}{2^{2\nu-\gamma+1}\Gamma(\nu+1)(\nu+1-\gamma)} \\
&\leq \frac{C}{\nu^{1/2+\gamma}}\left(\frac{\mathrm{e}}{4}\right)^\nu,
\end{aligned}$$

we have, using the estimate $\int_0^\nu |J_\nu(s)|\,\mathrm{d}s \leq C$ (see [7], Lemma 2.4),

$$\int_0^\nu \frac{\widetilde{J}_\nu(t)}{t^{1/4}}\,\mathrm{d}t = \int_0^{\nu/2} t^{1/4}J_\nu(t)\,\mathrm{d}t + \int_{\nu/2}^\nu t^{1/4}J_\nu(t)\,\mathrm{d}t \leq C\nu^{1/4}.$$

Therefore,

$$\begin{aligned}
\|T^1_\nu\|_2^2 &\leq C\sup_{u\in I} \frac{\nu^{1/4}}{u^{1/4}} \int_0^1 \frac{\widetilde{J}_\nu(uy)\chi_{[0,\,\nu]}(uy)}{y^{3/4}}\,\mathrm{d}y \\
&= C\sup_{u\in I} \frac{\nu^{1/4}}{u^{1/2}} \int_{[0,\,u]\cap[0,\,\nu]} J_\nu(s)s^{-1/4}\,\mathrm{d}s.
\end{aligned}$$

Assume first that $u > \nu/2$. Then

$$\frac{\nu^{1/4}}{u^{1/2}} \int_{[0,\,u]\cap[0,\,\nu]} \frac{J_\nu(s)}{s^{1/4}}\,\mathrm{d}s \leq \frac{C}{\nu^{1/4}}\left[\int_0^{\nu/2} + \int_{\nu/2}^\nu \frac{J_\nu(s)}{s^{1/4}}\,\mathrm{d}s\right] \leq \frac{C}{\nu^{1/2}} \leq C.$$

If instead $0 < u < \nu/2$, then

$$\begin{aligned}
&\frac{\nu^{1/4}}{u^{1/2}} \int_{[0,\,u]\cap[0,\,\nu]} \frac{J_\nu(s)}{s^{1/4}}\,\mathrm{d}s = \frac{\nu^{1/4}}{u^{1/2}} \int_0^u \frac{J_\nu(s)}{s^{1/4}}\,\mathrm{d}s \\
\leq\ & \frac{\nu^{1/4}}{u^{1/2}2^\nu\Gamma(\nu+1)} \int_0^u s^{\nu-1/4}ds \leq C\frac{u^{\nu+1/4}\nu^{-3/4}\mathrm{e}^\nu}{2^\nu\nu^{\nu+1/2}} \\
\leq\ & C\frac{\nu^{\nu+1/4}\nu^{-3/4}\mathrm{e}^\nu}{4^\nu\nu^{\nu+1/2}} = \frac{C}{\nu}\left(\frac{e}{4}\right)^\nu \leq C.
\end{aligned}$$

Thus $\|T^1_\nu\|_2^2 \leq C$. □

It is worth noting that in the study of T^1_ν we have not used the oscillation given by $\mathrm{e}^{\mathrm{i}t(r)s^2}$.

3. Boundedness of T_ν^2

Here we will use the following estimates on the Bessel functions: there exists a positive constant C such that if $s \in [\nu,\, \nu+\nu^{1/3}]$ then $|\widetilde{J}_\nu(s)| \le C\nu^{1/6}$, and if $s \in [\nu+\nu^{1/3},\, 2\nu]$ then

$$|\widetilde{J}_\nu(s)| \le C\frac{\nu^{1/4}}{(s-\nu)^{1/4}}.$$

These estimates are classical, but can be easily obtained from Lemma 1.3 too. We can now state the boundedness result for T_ν^2.

Proposition 3.1. *There exists a positive constant C such that for all $\nu \ge 1$, for all intervals I, for all functions $t(r)$ and for all $g \in L^2(I)$, we have*

$$\|T_\nu^2 g\|_{L^2([0,\,1])} \le C\|g\|_{L^2(I)}.$$

Proof. The absolute value of the kernel of T_ν^2 is

$$|k(s,\, r)| = |\widetilde{J}_\nu(sr)|\chi_{[\nu,\,\nu+\nu^{2/3}]}(sr)s^{-1/4}.$$

By Schur's lemma,

$$\|T_\nu^2\|_2^2 \le \sup_{u\in I}\int_0^1 |\widetilde{J}_\nu(uy)|\chi_{[\nu,\,\nu+\nu^{2/3}]}(uy)u^{-1/4}\int_I |\widetilde{J}_\nu(sy)|\chi_{[\nu,\,\nu+\nu^{2/3}]}(sy)s^{-1/4}\,\mathrm{d}s\,\mathrm{d}y.$$

The innermost integral is bounded above by

$$\begin{aligned}
&\frac{1}{y^{3/4}}\int_\nu^{\nu+\nu^{2/3}} |\widetilde{J}_\nu(t)|t^{-1/4}dt \le \frac{1}{y^{3/4}\nu^{1/4}}\int_\nu^{\nu+\nu^{2/3}} |\widetilde{J}_\nu(t)|\,\mathrm{d}t\\
\le\; & \frac{C}{y^{3/4}\nu^{1/4}}\left[\nu^{1/2} + \int_{\nu+\nu^{1/3}}^{\nu+\nu^{2/3}} \frac{\nu^{1/4}}{(t-\nu)^{1/4}}\,\mathrm{d}t\right]\\
\le\; & \frac{C}{y^{3/4}\nu^{1/4}}[\nu^{1/2}+\nu^{3/4}] \le C\frac{\nu^{1/2}}{y^{3/4}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|T_\nu^2\|_2^2 &\le C\nu^{1/2}\sup_{u\in I}\int_0^1 |\widetilde{J}_\nu(uy)|\chi_{[\nu,\,\nu+\nu^{2/3}]}(uy)u^{-1/4}y^{-3/4}\,\mathrm{d}y\\
&= C\nu^{1/2}\sup_{u\in I} u^{-1/2}\int_{[0,\,u]\cap[\nu,\,\nu+\nu^{2/3}]} |\widetilde{J}_\nu(t)|t^{-3/4}\,\mathrm{d}t\\
&\le C\nu^{-3/4}\int_\nu^{\nu+\nu^{2/3}} |\widetilde{J}_\nu(t)|\,\mathrm{d}t\\
&\le C\nu^{-3/4+3/4} = C.
\end{aligned}$$

□

Once more, in this proof we have not used the oscillation given by the exponential nor the one given by the Bessel function.

4. Boundedness of T_ν^3.

Proposition 4.1. *There exists a positive constant C such that for all $\nu \ge 1$, for all intervals I, for all functions $t(r)$ and for all $g \in L^2(I)$, we have*

$$\|T_\nu^3 g\|_{L^2([0,\,1])} \le C\|g\|_{L^2(I)}.$$

Proof. A trivial application of Cauchy-Schwartz's inequality yields

$$\begin{aligned}
\|T_\nu^3 g\|^2_{L^2([0,1])} &= \int_0^1 \left| \int_I \mathrm{e}^{\mathrm{i}t(r)s^2} r^{1/2} s^{1/4} h_\nu(rs) \chi_{[\nu+\nu^{2/3},\,2\nu]}(rs) g(s)\,\mathrm{d}s \right|^2 \mathrm{d}r \\
&\le \int_0^1 \int_I |h_\nu(rs)|^2 r s^{1/2} \chi_{[\nu+\nu^{2/3},\,2\nu]}(rs)\,\mathrm{d}s\,\mathrm{d}r\,\|g\|_2^2 \\
&\le \int_0^1 \int_{I'} |h_\nu(v)|^2 \frac{v^{1/2}}{r^{1/2}} \chi_{[\nu+\nu^{2/3},\,2\nu]}(v)\,\mathrm{d}v\,\mathrm{d}r\,\|g\|_2^2 \\
&\le C\|g\|_2^2 \int_{\nu+\nu^{2/3}}^{2\nu} |h_\nu(v)|^2 v^{1/2}\,\mathrm{d}v \\
&= C\nu^{3/2}\|g\|_2^2 \int_{1+\nu^{-1/3}}^{2} |h_\nu(\nu u)|^2 u^{1/2}\,\mathrm{d}u.
\end{aligned}$$

The estimate

$$|h_\nu(\nu u)|^2 \le C\left(\frac{1}{\nu^3(u^2-1)^{7/2}} + \frac{2}{\nu^{5/2}(u^2-1)^{7/4}u} + \frac{1}{\nu^2u^2}\right),$$

that holds for $u \in [1+\nu^{-2/3},\, 2]$, concludes the proof. □

5. Boundedness of T_ν^4.

We shall need the following technical lemma. Its proof is a simple application of the fundamental theorem of calculus.

Lemma 5.1. *Let I be an interval and $g \in \mathcal{C}^3(I)$ be such that $g'(u) \le 0$, $g''(u) \ge 0$ and $g'''(u) \le 0$ for all $u \in I$. Then for any u, $u_0 \in I$,*

(1) *if $u < u_0$, then $g(u) - g(u_0) \ge -g'(u_0)(u_0-u)$, and*
(2) *if $u > u_0$, then $g(u_0) - g(u) \ge -g'(u_0)(u-u_0) - \frac{1}{2}g''(u_0)(u-u_0)^2$.*

Proposition 5.2. *There exists a positive constant C such that for all $\nu \ge 1$, for all intervals I, for all functions $t(r)$ and for all $g \in L^2(I)$, we have*

$$\|T_\nu^4 g\|_{L^2([0,1])} \le C\|g\|_{L^2(I)}.$$

Proof. First write T_ν^4 as the sum of two operators, by means of the equality $\cos\theta = (\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})/2$,

$$\begin{aligned}
T_\nu^4 g(r) &= \sqrt{\frac{1}{2\pi}} \int_I \mathrm{e}^{\mathrm{i}t(r)s^2} \frac{r^{1/2}s^{1/4}\mathrm{e}^{\mathrm{i}\theta(rs)}}{(s^2r^2-\nu^2)^{1/4}} \chi_{[\nu+\nu^{2/3},\,2\nu]}(rs) g(s)\,\mathrm{d}s\, + \\
&\quad + \sqrt{\frac{1}{2\pi}} \int_I \mathrm{e}^{\mathrm{i}t(r)s^2} \frac{r^{1/2}s^{1/4}\mathrm{e}^{-\mathrm{i}\theta(rs)}}{(s^2r^2-\nu^2)^{1/4}} \chi_{[\nu+\nu^{2/3},\,2\nu]}(rs) g(s)\,\mathrm{d}s.
\end{aligned}$$

Observe that it is enough to study just one of the two above operators, as long as we obtain a result independent of the function $t(r)$, positive or negative. Let us then fix our attention on the one with the + sign in the exponential (call it just T). The operator TT^* has kernel

$$K(r,\rho) = \int_I \frac{\mathrm{e}^{\mathrm{i}[(t(r)-t(\rho))s^2+\theta(rs)-\theta(\rho s)]}(r\rho s)^{1/2}\chi_{[\nu+\nu^{2/3},\,2\nu]}(rs)\chi_{[\nu+\nu^{2/3},\,2\nu]}(\rho s)}{(r^2s^2-\nu^2)^{1/4}(\rho^2s^2-\nu^2)^{1/4}}\,\mathrm{d}s.$$

Let

$$\tilde\theta(x) = \theta(\nu x) = \nu\sqrt{x^2-1} - \nu\arccos(1/x) - \pi/4, \qquad x > 1.$$

Assuming $\rho < r$, calling $q = r/\rho$ and changing variables, $s = \nu u/\rho$, we have that the kernel $K(r, \rho)$ equals

$$\frac{\rho^{\beta-1/2}}{(r-\rho)^\beta}\left[\nu^{1/2}(q-1)^\beta \int_{I\cap[1+\nu^{-1/3},\, 2/q]} \frac{\mathrm{e}^{\mathrm{i}[-a\nu u^2/2+\tilde\theta(qu)-\tilde\theta(u)]}}{u^{1/2}(1-u^{-2})^{1/4}(1-q^{-2}u^{-2})^{1/4}}\,\mathrm{d}u\right],$$

where $a = -2\nu(t(r) - t(\rho))/\rho^2$ and $\beta \in [1/2, 1)$ will be fixed at our convenience ($\beta = 3/4$ will do). Since the function $\min(r, \rho)^{\beta-1/2}|r-\rho|^{-\beta}$ is integrable in $\rho \in [0, 1]$, uniformly in $r \in [0, 1]$, by Schur's lemma it is enough to show that the expression within brackets is uniformly bounded in $a \in \mathbb{R}$, $\nu \gg 1$, I any interval, and $q \in (1, 2)$ (for $q \geq 2$, the interval of integration becomes empty).

We introduce now some notation: for $u > 1$ call

$$\begin{aligned}
\psi(u) &= \frac{\nu^{1/2}(q-1)^\beta}{u^{1/2}(1-u^{-2})^{1/4}(1-q^{-2}u^{-2})^{1/4}} = \frac{\nu^{1/2}(q-1)^\beta u^{1/2}q^{1/2}}{(u^2-1)^{1/4}(q^2u^2-1)^{1/4}},\\
\phi(u) &= -a\nu u^2/2 + \tilde\theta(qu) - \tilde\theta(u),\\
\eta &= -\log_\nu(q-1),
\end{aligned}$$

so that $q = 1 + \nu^{-\eta}$, and the required uniformity in $q \in (1, 2)$ is moved to the same one for $\eta > 0$. Next observe that for $\eta \geq 1/(2\beta)$, the result is easily obtained since

$$\left|\int_{I\cap[1+\nu^{-1/3},\, 2/q]} \mathrm{e}^{\mathrm{i}\phi(u)}\psi(u)\,\mathrm{d}u\right| \leq \int_1^2 \psi(u)\,\mathrm{d}u \leq C\int_1^2 \frac{\nu^{1/2-\eta\beta}}{(u-1)^{1/2}}\,\mathrm{d}u \leq C.$$

Let us assume then that $0 < \eta < 1/(2\beta)$. This is the point where we start using the oscillatory term in the estimation of our integral. Since we want to use Van der Corput's lemma, we need to study the function ϕ'. Note that

$$\phi'(u) = \nu\left(\sqrt{q^2-u^{-2}} - \sqrt{1-u^{-2}} - au\right) = \nu(f(u) - au),$$

where f is implicitly defined by the above equality. Let us begin by considering only those values of a for which there is a zero of ϕ' in the interval $[1+\nu^{-1/3}, 2/q]$. Thus, parametrize a in such a way that this zero is $1+\nu^{-\gamma}$, with $\gamma \in [0, 1/3]$. This way, the required uniformity in the parameter a is moved to the parameter γ. For further reference, observe that

$$a = \frac{\sqrt{q^2(1+\nu^{-\gamma})^2-1} - \sqrt{(1+\nu^{-\gamma})^2-1}}{(1+\nu^{-\gamma})^2}.$$

Let $u \in [1+\nu^{-1/3}, 2/q]$. One can easily see that f satisfies all the hypotheses of Lemma 5.1. Thus, recalling that $\phi'(u) = \nu(f(u) - au)$, we may deduce that if $u < 1 + \nu^{-\gamma}$ then

$$|\phi'(u)| \geq \nu(1+\nu^{-\gamma}-u)\left(a - f'(1+\nu^{-\gamma})\right), \tag{5.1}$$

whereas if $u > 1 + \nu^{-\gamma}$

$$|\phi'(u)| \geq \nu(u-1-\nu^{-\gamma})\left(a - f'(1+\nu^{-\gamma}) - \frac{1}{2}f''(1+\nu^{-\gamma})(u-1-\nu^{-\gamma})\right). \tag{5.2}$$

Define $\xi = \min(\eta, \gamma)$ and

$$\delta = \frac{1}{2} - \eta\beta + \frac{\gamma}{4} + \frac{\xi}{4}.$$

Observe that (η, γ) may vary in the rectangle $\mathcal{R} = (0, 1/(2\beta)) \times [0, 1/3]$. Divide $\mathcal{R}$ into two regions, $\mathcal{F} = \{(\eta, \gamma) \in \mathcal{R}\colon \delta \geq \gamma\}$ and $\mathcal{G} = \mathcal{R} \setminus \mathcal{F}$.

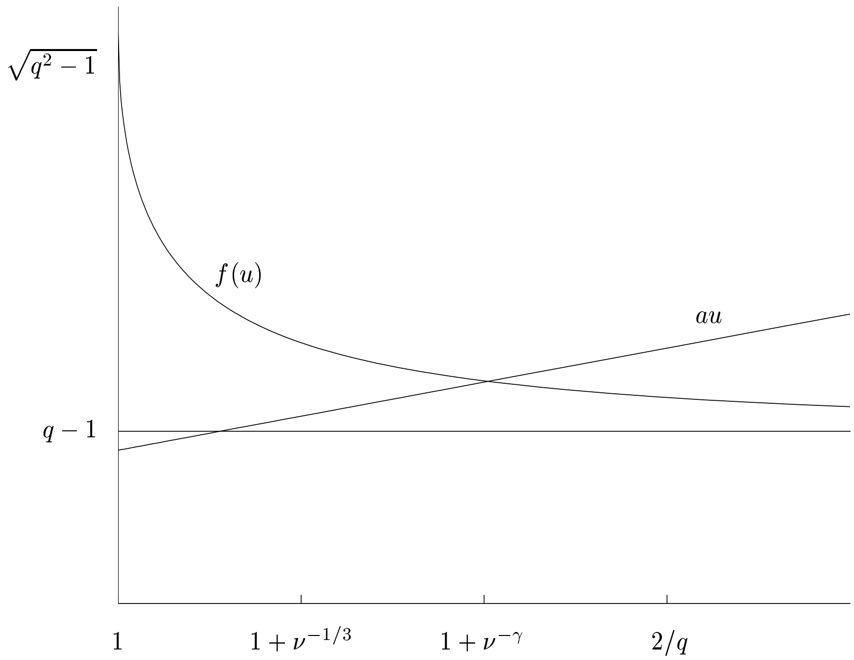

FIGURE 1. The curves $f(u)$ and au.

Consider first the case $(\eta,\gamma)\in\mathcal{F}$. Divide the interval $[1+\nu^{-1/3},\, 2/q]$ into the union of four subintervals (defined to be empty when the left endpoint happens to be bigger than the right endpoint):

$$\begin{aligned}
\mathcal{A}_1 &= [1+\nu^{-1/3},\, 1+\nu^{-\gamma}/10],\\
\mathcal{A}_2 &= [1+\nu^{-\gamma}/10,\, 1+\nu^{-\gamma}-\nu^{-\delta}/N],\\
\mathcal{A}_3 &= [1+\nu^{-\gamma}-\nu^{-\delta}/N,\, 1+\nu^{-\gamma}+\nu^{-\delta}/N],\\
\mathcal{A}_4 &= [1+\nu^{-\gamma}+\nu^{-\delta}/N,\, 2/q],
\end{aligned}$$

where N is a large number that will be fixed at our convenience. The interval $\mathcal{A}_3$ is a neighborhood of the zero of ϕ', where the oscillation vanishes. The best we can do here is then to estimate the corresponding integral with the magnitude of the integrand:

$$\begin{aligned}
\left|\int_{I\cap\mathcal{A}_3} \mathrm{e}^{\mathrm{i}\phi(u)}\psi(u)\,\mathrm{d}u\right| &\leq \int_{1+\nu^{-\gamma}-\nu^{-\delta}/N}^{1+\nu^{-\gamma}+\nu^{-\delta}/N} \psi(u)\,\mathrm{d}u \leq \frac{2}{N\nu^{\delta}}\,\psi\left(1+\frac{1}{2\nu^{\gamma}}\right)\\
&\leq C\frac{1}{\nu^{\delta}}\frac{\nu^{1/2-\eta\beta}}{\nu^{-\gamma/4}((1+\nu^{-\eta})(1+\nu^{-\gamma}/2)-1)^{1/4}}\\
&\leq C\frac{1}{\nu^{\delta}}\frac{\nu^{1/2-\eta\beta+\gamma/4}}{(\nu^{-\eta}+\nu^{-\gamma})^{1/4}} \leq C\frac{\nu^{\delta}}{\nu^{\delta}} \leq C.
\end{aligned}$$

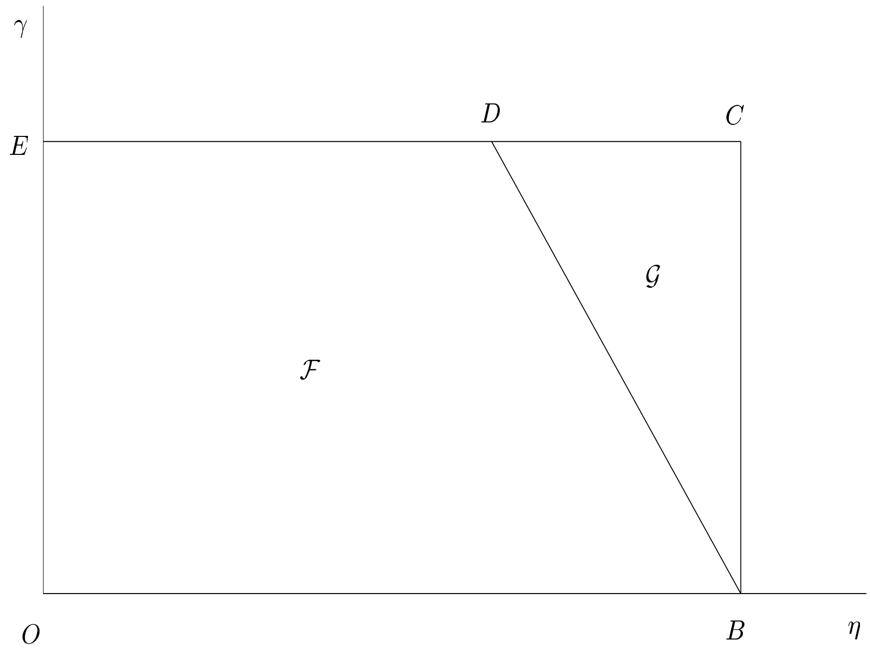

FIGURE 2. The sets $\mathcal{F}$ and $\mathcal{G}$, where $B = (1/(2\beta),\, 0)$, $C = (1/(2\beta),\, 1/3)$, $D = (1/(3\beta),\, 1/3)$ and $E = (0,\, 1/3)$.

For $\mathcal{A}_1$, we can use Van der Corput's lemma. Thus

$$\left|\int_{I\cap\mathcal{A}_1} \mathrm{e}^{\mathrm{i}\phi(u)}\psi(u)\,\mathrm{d}u\right| \leq C\,\frac{\psi(1+\nu^{-1/3})}{\phi'(1+\nu^{-\gamma}/10)}.$$

Observe that

$$\psi(1+\nu^{-1/3}) \leq C\frac{\nu^{1/2-\eta\beta}}{\nu^{-1/12}(\nu^{-1/3}+\nu^{-\eta})^{1/4}} \leq C\,\nu^{7/12-\eta\beta+\zeta/4},$$

where $\zeta = \min(\eta,\, 1/3)$. As for ϕ', using (5.1) we have that

$$\begin{aligned}
|\phi'(1+\nu^{-\gamma}/10)| &\geq \nu(1+\nu^{-\gamma}-1-\nu^{-\gamma}/10)\left(a-f'(1+\nu^{-\gamma})\right)\\
&\geq \frac{9}{10}\,\nu^{1-\gamma}\left(-f'(1+\nu^{-\gamma})\right)\\
&= \frac{9}{10}\,\nu^{1-\gamma}\left(-\frac{(1+\nu^{-\gamma})^{-2}}{\sqrt{q^2(1+\nu^{-\gamma})^2-1}}+\frac{(1+\nu^{-\gamma})^{-2}}{\sqrt{(1+\nu^{-\gamma})^2-1}}\right)\\
&\geq C\,\nu^{1-\gamma}\,\frac{\sqrt{(1+\nu^{-\eta})^2(1+\nu^{-\gamma})^2-1}-\sqrt{(1+\nu^{-\gamma})^2-1}}{\sqrt{(1+\nu^{-\eta})^2(1+\nu^{-\gamma})^2-1}\sqrt{(1+\nu^{-\gamma})^2-1}}\\
&\geq C\nu^{1-\gamma/2-\eta+\xi}.
\end{aligned}$$

Thus

$$\begin{aligned}\left|\int_{I\cap\mathcal{A}_1} \mathrm{e}^{\mathrm{i}\phi(u)}\psi(u)\,\mathrm{d}u\right| &\le C\frac{\nu^{7/12-\eta\beta+\zeta/4}}{\nu^{1-\gamma/2-\eta+\xi}}\\ &\le C\nu^{-5/12+\eta(1-\beta)+\gamma/2+\zeta/4-\xi} \le C\nu^{-1/6+(1-\beta)/(2\beta)} \le C,\end{aligned}$$

if $\beta \ge 3/4$.

Let us now consider $\mathcal{A}_2$. Once again, using Van der Corput's lemma,

$$\left|\int_{\mathcal{A}_2\cap I} \mathrm{e}^{\mathrm{i}\phi(u)}\psi(u)\,\mathrm{d}u\right| \le C\frac{\psi(1+\nu^{-\gamma}/10)}{\phi'(1+\nu^{-\gamma}-\nu^{-\delta}/N)}.$$

Proceeding as in the previous case, we see that

$$\psi(1+\nu^{-\gamma}/10) \le C\frac{\nu^{1/2-\eta\beta}}{\nu^{-\gamma/4}(\nu^{-\gamma}+\nu^{-\eta})^{1/4}} \le C\,\nu^{1/2-\eta\beta+\gamma/4+\xi/4},$$

and that

$$\begin{aligned}|\phi'(1+\nu^{-\gamma}-\nu^{-\delta}/N)| &\ge \nu(1+\nu^{-\gamma}-1-\nu^{-\gamma}+\nu^{-\delta}/N)\left(a-f'(1+\nu^{-\gamma})\right)\\ &\ge \frac{\nu^{1-\delta}}{N}\left(-f'(1+\nu^{-\gamma})\right)\\ &\ge C\nu^{1-\delta-\eta+\gamma/2+\xi}\\ &= C\nu^{1/2-\eta(1-\beta)+\gamma/4+3\xi/4}.\end{aligned}$$

Therefore

$$\begin{aligned}\left|\int_{I\cap\mathcal{A}_2} \mathrm{e}^{\mathrm{i}\phi(u)}\psi(u)\,\mathrm{d}u\right| &\le C\frac{\nu^{1/2-\eta\beta+\gamma/4+\xi/4}}{\nu^{1/2-\eta(1-\beta)+\gamma/4+3\xi/4}}\\ &\le C\nu^{-(2\beta-1)\eta-\xi/2} \le C,\end{aligned}$$

if $\beta \ge 1/2$.

Let us now move to the study of the interval $\mathcal{A}_4$. Using Van der Corput's lemma, we have

$$\left|\int_{\mathcal{A}_4\cap I} \mathrm{e}^{\mathrm{i}\phi(u)}\psi(u)\,\mathrm{d}u\right| \le C\frac{\psi(1+\nu^{-\gamma}+\nu^{-\delta}/N)}{|\phi'(1+\nu^{-\gamma}+\nu^{-\delta}/N)|}.$$

As usual, we see that

$$\psi(1+\nu^{-\gamma}+\nu^{-\delta}/N) \le C\frac{\nu^{1/2-\eta\beta}}{\nu^{-\gamma/4}(\nu^{-\gamma}+\nu^{-\eta})^{1/4}} \le C\,\nu^{1/2-\eta\beta+\gamma/4+\xi/4},$$

while using (5.2), we see that

$$\begin{aligned}|\phi'(1+\nu^{-\gamma}+\nu^{-\delta}/N)| &\ge \nu\,\frac{1}{N\nu^{\delta}}\left(a-f'(1+\nu^{-\gamma})-\frac{1}{2}f''(1+\nu^{-\gamma})\frac{1}{N\nu^{\delta}}\right)\\ &\ge \frac{\nu^{1-\delta}}{N}\left(-f'(1+\nu^{-\gamma})-\frac{1}{2}f''(1+\nu^{-\gamma})\frac{1}{N\nu^{\delta}}\right).\end{aligned}$$

We already have the estimate $-f'(1+\nu^{-\gamma}) \ge C\nu^{\gamma/2+\xi-\eta}$. We shall now show that there is a positive constant C such that

$$|f''(1+\nu^{-\gamma})| \le C\nu^{3\gamma/2+\xi-\eta}. \tag{5.3}$$

In the following computations, we will call $u_0 = 1+x = 1+\nu^{-\gamma}$, $q = 1+y = 1+\nu^{-\eta}$, with $x, y \in [0, 1]$, and $z = \max(x, y)$. Thus

$$|f''(u_0)| = \frac{(3u_0^2-2)(q^2u_0^2-1)^{3/2} - (3q^2u_0^2-2)(u_0^2-1)^{3/2}}{u_0^3(u_0^2-1)^{3/2}(q^2u_0^2-1)^{3/2}} =$$
$$\frac{(3u_0^2-2)^2(q^2u_0^2-1)^3 - (3q^2u_0^2-2)^2(u_0^2-1)^3}{u_0^3(u_0^2-1)^{3/2}(q^2u_0^2-1)^{3/2}\left((3u_0^2-2)(q^2u_0^2-1)^{3/2} + (3q^2u_0^2-2)(u_0^2-1)^{3/2}\right)}.$$

The numerator of the above expression is a polynomial in x and y, sum of monomials of degrees 3 to 16, none of which is of the form x^j for any j. Therefore this numerator is bounded above in absolute value by

$$Cy(x^2+xy+y^2) \le Cyz^2.$$

On the other hand, the denominator is bounded below in absolute value by

$$Cx^{3/2}(x+y)^{3/2}((x+y)^{3/2}+x^{3/2}) \ge Cx^{3/2}z^3.$$

It follows that

$$|f''(u_0)| \le C\frac{yz^2}{x^{3/2}z^3} = C\nu^{3\gamma/2+\xi-\eta},$$

as desired. Thus we may deduce that

$$\begin{aligned}
|\phi'(1+\nu^{-\gamma}+\nu^{-\delta}/N)| &\ge \frac{C}{N}\nu^{1-\delta}\left(\nu^{\gamma/2+\xi-\eta} - \frac{C\nu^{3\gamma/2+\xi-\eta}}{N\nu^{\delta}}\right)\\
&\ge \frac{C}{N}\nu^{1-\delta}\nu^{\gamma/2+\xi-\eta}(1-\frac{C}{N}\nu^{\gamma-\delta})\\
&\ge C\nu^{1-\delta+\gamma/2+\xi-\eta}\\
&\ge C\nu^{1/2+\gamma/4+3\xi/4-\eta(1-\beta)},
\end{aligned}$$

if we take N big enough (recall we are in the case $\delta \ge \gamma$). We may now conclude

$$\left|\int_{\mathcal{A}_4\cap I} \mathrm{e}^{\mathrm{i}\phi(u)}\psi(u)\,\mathrm{d}u\right| \le C\frac{\nu^{1/2-\eta\beta+\gamma/4+\xi/4}}{\nu^{1/2+\gamma/4+3\xi/4-\eta(1-\beta)}} \le C\nu^{-\eta(2\beta-1)-\xi/2} \le C,$$

if $\beta \ge 1/2$.

It remains to study the case $(\eta, \gamma) \in \mathcal{G}$, that is $\delta < \gamma$. Observe that this implies $\gamma \le \eta$ and therefore $\xi = \gamma$. Divide the interval $[1+\nu^{-1/3}, 2/q]$ into the union of three subintervals (defined to be empty when the left endpoint happens to be bigger than the right endpoint):

$$\begin{aligned}
\mathcal{A}_1 &= [1+\nu^{-1/3},\, 1+\nu^{-\gamma}/10],\\
\mathcal{A}_2 &= [1+\nu^{-\gamma}/10,\, 1+2\nu^{-1/2+\eta\beta-\gamma/2}],\\
\mathcal{A}_3 &= [1+2\nu^{-1/2+\eta\beta-\gamma/2},\, 2/q],
\end{aligned}$$

The interval $\mathcal{A}_2$ is a neighborhood of the zero of ϕ', where the oscillation vanishes, so we estimate the associated integral with the magnitude of the integrand:

$$\begin{aligned}
\left|\int_{I\cap\mathcal{A}_2} \mathrm{e}^{\mathrm{i}\phi(u)}\psi(u)\,\mathrm{d}u\right| &\leq \int_{1+\nu^{-\gamma}/10}^{1+2\nu^{-1/2+\eta\beta-\gamma/2}} \psi(u)\,\mathrm{d}u \\
&\leq 2\nu^{-1/2+\eta\beta-\gamma/2}\,\psi\left(1+\nu^{-\gamma}/10\right) \\
&\leq C\frac{\nu^{-1/2+\eta\beta-\gamma/2}\nu^{1/2-\eta\beta}}{\nu^{-\gamma/4}((1+\nu^{-\eta})(1+\nu^{-\gamma}/10)-1)^{1/4}} \\
&\leq C\frac{\nu^{-\gamma/2}}{\nu^{-\gamma/2}} \leq C.
\end{aligned}$$

The study of $\mathcal{A}_1$ is exactly the same as in the case $\gamma \leq \delta$, thus we do not repeat it.

As for $\mathcal{A}_3$, we use Van der Corput's lemma, obtaining

$$\left|\int_{\mathcal{A}_3\cap I} \mathrm{e}^{\mathrm{i}\phi(u)}\psi(u)\,\mathrm{d}u\right| \leq C\frac{\psi(1+2\nu^{-1/2+\eta\beta-\gamma/2})}{|\phi'(1+2\nu^{-1/2+\eta\beta-\gamma/2})|}.$$

Since a is positive and f is decreasing and $f(1+\nu^{-\gamma}) = a(1+\nu^{-\gamma})$, we may say that

$$\begin{aligned}
|\phi'(1+2\nu^{-1/2+\eta\beta-\gamma/2})| &\geq \nu(a(1+2\nu^{-1/2+\eta\beta-\gamma/2}) - f(1+\nu^{-\gamma})) \\
&= \nu a(2\nu^{-1/2+\eta\beta-\gamma/2} - \nu^{-\gamma}) \geq \nu a\nu^{-1/2+\eta\beta-\gamma/2} \\
&\geq \nu^{1/2+\eta\beta-\gamma/2}\frac{q^2-1}{\sqrt{q^2(1+\nu^{-\gamma})^2-1}+\sqrt{(1+\nu^{-\gamma})^2-1}} \\
&\geq C\nu^{1/2+\eta\beta-\gamma/2}\frac{\nu^{-\eta}}{(\nu^{-\gamma}+\nu^{-\eta})^{1/2}+\nu^{-\gamma/2}} \\
&\geq C\nu^{1/2+\eta\beta-\gamma/2}\nu^{-\eta+\gamma/2} = C\nu^{1/2+\eta\beta-\eta}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\psi(1+2\nu^{-1/2+\eta\beta-\gamma/2}) &\leq C\frac{\nu^{1/2-\eta\beta}}{\nu^{-1/8+\eta\beta/4-\gamma/8}(\nu^{-\eta}+\nu^{-1/2+\eta\beta-\gamma/2})^{1/4}} \\
&\leq C\frac{\nu^{1/2-\eta\beta}}{\nu^{-1/4+\eta\beta/2-\gamma/4}} = C\nu^{3/4-3\eta\beta/2+\gamma/4}.
\end{aligned}$$

Therefore,

$$\left|\int_{\mathcal{A}_3\cap I} \mathrm{e}^{\mathrm{i}\phi(u)}\psi(u)\,\mathrm{d}u\right| \leq C\,\frac{\nu^{3/4-3\eta\beta/2+\gamma/4}}{\nu^{1/2+\eta\beta-\eta}} = C\nu^{1/4-5\eta\beta/2+\gamma/4+\eta} \leq C,$$

if $(\eta,\,\gamma) \in \mathcal{G}$, and $\beta \geq 2/3$.

It remains to study the boundedness of the integral

$$\left|\int_{I\cap[1+\nu^{-1/3},\,2/q]} \mathrm{e}^{\mathrm{i}\phi(u)}\psi(u)\,\mathrm{d}u\right|$$

for the values of a for which ϕ' has no zeros in $[1+\nu^{-1/3},\,2/q]$. Call a_0 and a_1 the values of a for which the zero of ϕ' is $1+\nu^{-1/3}$ and $2/q$, respectively. Geometrically,

it is clear that for any fixed $u \in [1+\nu^{-1/3},\, 2/q]$, the value of $|\phi'(u)|$ grows as a goes from a_0 to ∞, and decreases as a goes from $-\infty$ to a_1, while $\psi(u)$ stays unchanged. Thus, all the estimates we obtained for a_0 using Van der Corput's lemma or simply the magnitude of ψ, remain true for any $a \geq a_0$, and those we obtained for a_1 remain true for any $a \leq a_1$. This concludes the proof. □

6. Boundedness of T_ν^5.

Proposition 6.1. *There exists a positive constant C such that for all $\nu \geq 1$, for all intervals I, for all functions $t(r)$ and for all $g \in L^2(I)$, we have*

$$\|T_\nu^5 g\|_{L^2([0,\,1])} \leq C\|g\|_{L^2(I)}.$$

Proof. The kernel of the operator $T_\nu^5(T_\nu^5)^*$ is

$$L(r,\,\rho) = \int_I \mathrm{e}^{\mathrm{i}(t(r)-t(\rho))s^2}\widetilde{h}_\nu(rs)\widetilde{h}_\nu(\rho s)\chi_{[2\nu,\,\infty)}(rs)\chi_{[2\nu,\,\infty)}(\rho s)s^{-1/2}\,\mathrm{d}s,$$

Thus, using Lemma 1.3,

$$|L(r,\,\rho)| \leq \int_{\frac{2\nu}{\min(r,\,\rho)}}^{\infty} |\widetilde{h}_\nu(rs)\widetilde{h}_\nu(\rho s)|s^{-1/2}\,\mathrm{d}s \leq C\int_{\frac{2\nu}{\min(r,\,\rho)}}^{\infty} \frac{s^{-3/2}}{\sqrt{r\rho}}\,\mathrm{d}s \leq \frac{C\sqrt{\min(r,\,\rho)}}{\sqrt{\nu r\rho}}.$$

Since

$$\int_0^1 |L(r,\,\rho)|\,\mathrm{d}r \leq \int_0^1 \frac{C\sqrt{\min(r,\,\rho)}}{\sqrt{\nu r\rho}}\,\mathrm{d}r = \frac{C}{\sqrt{\nu\rho}}\int_0^\rho \mathrm{d}r + \frac{C}{\sqrt{\nu}}\int_\rho^1 \frac{\mathrm{d}r}{\sqrt{r}} = \frac{C}{\sqrt{\nu}}(2-\sqrt{\rho})$$

is uniformly bounded in $\rho \in [0,\,1]$, by Schur's lemma the operators $T_\nu^5(T_\nu^5)^*$ are uniformly bounded, and so are the T_ν^5's. □

7. Boundedness of T_ν^6.

Proposition 7.1. *There exists a positive constant C such that for all $\nu \geq 1$, for all intervals I, for all functions $t(r)$ and for all $g \in L^2(I)$, we have*

$$\|T_\nu^6 g\|_{L^2([0,\,1])} \leq C\|g\|_{L^2(I)}.$$

Proceeding as for T_ν^4, write T_ν^6 as the sum of two operators, by means of the equality $\cos\theta = (\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta})/2$,

$$\begin{aligned} T_\nu^6 g(r) \quad &= \quad \sqrt{\frac{1}{2\pi}}\int_I \mathrm{e}^{\mathrm{i}t(r)s^2}\frac{r^{1/2}s^{1/4}\mathrm{e}^{\mathrm{i}\theta(rs)}}{(s^2r^2-\nu^2)^{1/4}}\chi_{[2\nu,\,\infty)}(rs)g(s)\,\mathrm{d}s\,+ \\ &\quad +\sqrt{\frac{1}{2\pi}}\int_I \mathrm{e}^{\mathrm{i}t(r)s^2}\frac{r^{1/2}s^{1/4}\mathrm{e}^{-\mathrm{i}\theta(rs)}}{(s^2r^2-\nu^2)^{1/4}}\chi_{[2\nu,\,\infty)}(rs)g(s)\,\mathrm{d}s. \end{aligned}$$

Once again, it is enough to study just one of these two operators, for example the one with the + sign in the exponential (call it just T). The operator TT^* has kernel

$$K(r,\,\rho) = \int_I \frac{\mathrm{e}^{\mathrm{i}[(t(r)-t(\rho))s^2+\theta(rs)-\theta(\rho s)]}r^{1/2}\rho^{1/2}s^{1/2}\chi_{[2\nu,\,\infty)}(rs)\chi_{[2\nu,\,\infty)}(\rho s)}{(r^2s^2-\nu^2)^{1/4}(\rho^2s^2-\nu^2)^{1/4}}\,\mathrm{d}s.$$

Assuming $\rho < r$, calling $p = (r-\rho)/\rho$ and $\sigma = p\nu$, and changing variables, $s = u/(r-\rho)$, we have the kernel

$$\frac{1}{(r-\rho)^{1/2}}\int_{I\cap[2\sigma,\,\infty)} \frac{\mathrm{e}^{\mathrm{i}[-au^2/2+\theta((p+1)u/p)-\theta(u/p)]}}{u^{1/2}(1-\sigma^2(p+1)^{-2}u^{-2})^{1/4}(1-\sigma^2u^{-2})^{1/4}}\,\mathrm{d}u,$$

where $a = -2(t(r) - t(\rho))/(r - \rho)^2$. Since the function $|r - \rho|^{-1/2}$ is integrable in r, uniformly in ρ, by Schur's lemma it is enough to show that the integral is uniformly bounded in the interval I, in $p > 0$, in $\sigma > 0$, and in $a \in \mathbb{R}$. Let us call

$$\begin{aligned} \phi(u) &= -\frac{a}{2}u^2 + \theta\left(\frac{p+1}{p}u\right) - \theta\left(\frac{u}{p}\right) \\ \psi(u) &= \frac{1}{u^{1/2}\left(1 - \frac{\sigma^2}{(p+1)^2u^2}\right)^{1/4}\left(1 - \frac{\sigma^2}{u^2}\right)^{1/4}}. \end{aligned}$$

Observe that

$$\phi'(u) = -au + \frac{(p+2)u}{\sqrt{(p+1)^2u^2 - \sigma^2} + \sqrt{u^2 - \sigma^2}} = -au + f(u),$$

(note that here "f" indicates a different function from the one in section 5) and that the function ψ is decreasing with

$$\psi(u) \le \frac{1}{\sqrt{u - \sigma}}.$$

Note that, since ϕ' is the difference of a concave up function and a linear function, ϕ'' is the difference of an increasing function and a constant. Hence, ϕ'' is increasing and therefore it changes sign at most once. By assuming that the interval I is contained in an interval where ϕ'' has constant sign, we can apply Van der Corput's lemma to

$$\left|\int_{I\cap[2\sigma,\,\infty)} \mathrm{e}^{\mathrm{i}\phi(u)}\psi(u)\,\mathrm{d}u\right|.$$

In order to do it, we need to study the function ϕ'. As usual, we consider only those values of a for which there is a zero of ϕ' in the interval $[2\sigma, \infty)$, that is

$$0 < a \le \frac{\sqrt{4(p+1)^2 - 1} - \sqrt{3}}{4\sigma p}.$$

Assume first that $\sigma \ge 1$. Let us parametrize a in such a way that the zero of ϕ' is $\sigma + \sigma^\gamma$, with $\gamma \ge 1$. This gives

$$a = \frac{(p+2)}{\sqrt{(p+1)^2(\sigma + \sigma^\gamma)^2 - \sigma^2} + \sqrt{(\sigma + \sigma^\gamma)^2 - \sigma^2}}.$$

In this way, the required uniformity in the parameter a is equivalent to the uniformity in the parameter γ. In order to apply Van der Corput's lemma we need to

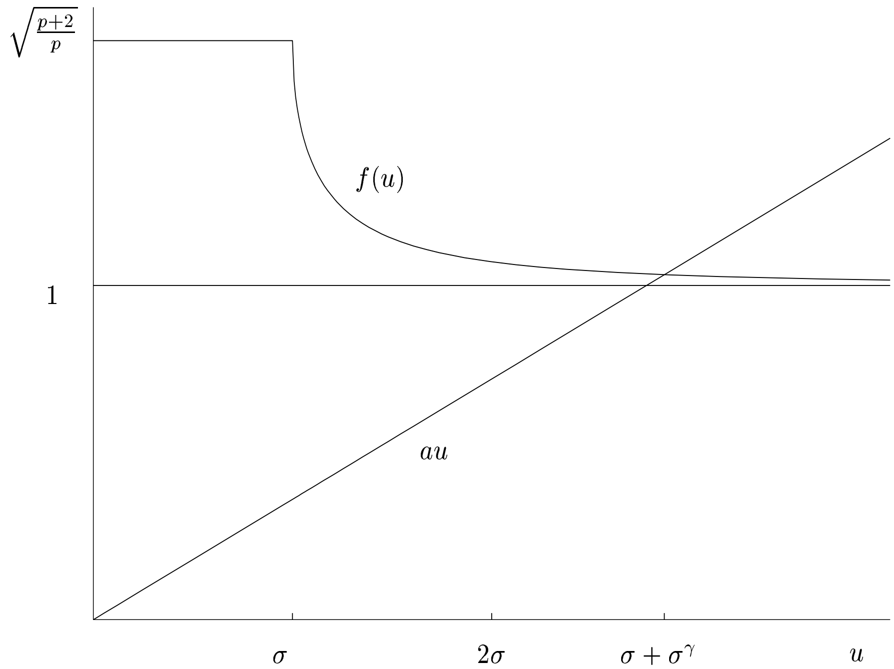

FIGURE 3. The curves $f(u)$ and au.

estimate $|\phi'|$ from below. Observe that

$$\begin{aligned}
|\phi'(u)| &= \left|-\frac{(p+2)u}{\sqrt{(p+1)^2(\sigma+\sigma^\gamma)^2-\sigma^2}+\sqrt{(\sigma+\sigma^\gamma)^2-\sigma^2}}+\right.\\
&\quad \left.+\frac{(p+2)u}{\sqrt{(p+1)^2u^2-\sigma^2}+\sqrt{u^2-\sigma^2}}\right|\\
&= \left|\frac{(p+2)u}{(\sqrt{(p+1)^2u^2-\sigma^2}+\sqrt{u^2-\sigma^2})}\times\right.\\
&\quad \times\frac{1}{(\sqrt{(p+1)^2(\sigma+\sigma^\gamma)^2-\sigma^2}+\sqrt{(\sigma+\sigma^\gamma)^2-\sigma^2})}\times\\
&\quad \times\left(\frac{(p+1)^2((\sigma+\sigma^\gamma)^2-u^2)}{\sqrt{(p+1)^2(\sigma+\sigma^\gamma)^2-\sigma^2}+\sqrt{(p+1)^2u^2-\sigma^2}}+\right.\\
&\quad \left.\left.+\frac{(\sigma+\sigma^\gamma)^2-u^2}{\sqrt{(\sigma+\sigma^\gamma)^2-\sigma^2}+\sqrt{u^2-\sigma^2}}\right)\right|\\
&\geq \left|\frac{(p+2)u}{((p+1)u+u)((p+1)(\sigma+\sigma^\gamma)+(\sigma+\sigma^\gamma))}\times\right.\\
&\quad \left.\times\left(\frac{(p+1)^2((\sigma+\sigma^\gamma)^2-u^2)}{(p+1)(\sigma+\sigma^\gamma)+(p+1)u}+\frac{(\sigma+\sigma^\gamma)^2-u^2}{\sigma+\sigma^\gamma+u}\right)\right|\\
&= \left|\frac{\sigma+\sigma^\gamma-u}{\sigma+\sigma^\gamma}\right|.
\end{aligned}$$

Next divide the interval $[2\sigma,\, \infty)$ into four subintervals, given by the following partition

$$\begin{aligned} u_1 &= 2\sigma, \\ u_2 &= \max(2\sigma,\, \sigma + \sigma^\gamma/2), \\ u_3 &= \max(u_2,\, \sigma + \sigma^\gamma - \sigma^{\gamma/2}), \\ u_4 &= \sigma + \sigma^\gamma + \sigma^{\gamma/2}, \end{aligned}$$

and study each case separately. Applying Van der Corput's lemma and using the above estimates for ψ and ϕ', we obtain that when $[u_1,\, u_2]$ is non-degenerate,

$$\left| \int_{[u_1,\, u_2]\cap I} \mathrm{e}^{\mathrm{i}\phi(u)} \psi(u)\, \mathrm{d}u \right| \le C \frac{\psi(2\sigma)}{|\phi'(\sigma + \sigma^\gamma/2)|} \le C \frac{\sigma + \sigma^\gamma}{\sqrt{\sigma}\sigma^\gamma/2} \le \frac{C}{\sqrt{\sigma}} \le C.$$

On the other hand, when $[u_2,\, u_3]$ is non-degenerate,

$$\left| \int_{[u_2,\, u_3]\cap I} \mathrm{e}^{\mathrm{i}\phi(u)} \psi(u)\, \mathrm{d}u \right| \le C \frac{\psi(\sigma + \sigma^\gamma/2)}{|\phi'(\sigma + \sigma^\gamma - \sigma^{\gamma/2})|} \le C \frac{\sigma + \sigma^\gamma}{\sqrt{\sigma^\gamma/2}\sigma^{\gamma/2}} \le C.$$

As for $[u_3,\, u_4]$, we estimate it using the size of $\psi(u)$:

$$\begin{aligned} \left| \int_{[u_3,\, u_4]\cap I} \mathrm{e}^{\mathrm{i}\phi(u)} \psi(u)\, \mathrm{d}u \right| &\le \int_{u_3}^{\sigma+\sigma^\gamma+\sigma^{\gamma/2}} \psi(u)\, \mathrm{d}u \le 2\sigma^{\gamma/2}\psi(u_3) \\ &\le 2\sigma^{\gamma/2}\psi(\sigma + \sigma^\gamma/2) \le \frac{2\sigma^{\gamma/2}}{\sqrt{\sigma^\gamma/2}} \le C. \end{aligned}$$

Finally, using Van der Corput's lemma again,

$$\left| \int_{[u_4,\, \infty]\cap I} \mathrm{e}^{\mathrm{i}\phi(u)} \psi(u)\, \mathrm{d}u \right| \le C \frac{\psi(\sigma + \sigma^\gamma + \sigma^{\gamma/2})}{|\phi'(\sigma + \sigma^\gamma + \sigma^{\gamma/2})|} \le C \frac{\sigma + \sigma^\gamma}{\sqrt{\sigma^\gamma + \sigma^{\gamma/2}}\sigma^{\gamma/2}} \le C.$$

This concludes the case $\sigma \ge 1$. As for the remaining case, $0 < \sigma \le 1$, we impose that the zero of ϕ' is $\sigma + \sigma^\gamma$, with $\gamma \le 1$ (when γ grows from $-\infty$ to 1, σ^γ decreases from ∞ to σ). Just as before, we have the following estimates for ϕ' and ψ

$$\begin{aligned} \psi(u) &\le \frac{1}{\sqrt{u - \sigma}}, \\ |\phi'(u)| &\ge \frac{|\sigma + \sigma^\gamma - u|}{\sigma + \sigma^\gamma} \ge \frac{|\sigma + \sigma^\gamma - u|}{2\sigma^\gamma}. \end{aligned}$$

Suppose $0 \le \gamma \le 1$. Then

$$\left| \int_{[2\sigma,\, 3]\cap I} \mathrm{e}^{\mathrm{i}\phi(u)} \psi(u)\, \mathrm{d}u \right| \le \int_{2\sigma}^{3} \psi(u)\, \mathrm{d}u \le C,$$

and by Van der Corput's lemma,

$$\left| \int_{[3,\, \infty]\cap I} \mathrm{e}^{\mathrm{i}\phi(u)} \psi(u)\, \mathrm{d}u \right| \le C \frac{\psi(3)}{|\phi'(3)|} \le C \frac{2\sigma^\gamma}{\sqrt{2}(3 - \sigma - \sigma^\gamma)} \le C\sigma^\gamma \le C.$$

If instead $\gamma < 0$, then we divide the interval $[2\sigma, \infty)$ into five subintervals, given by the following partition

$$\begin{aligned} u_1 &= 2\sigma, \\ u_2 &= 3, \\ u_3 &= \max(3, \sigma + \sigma^\gamma/2), \\ u_4 &= \max(u_3, \sigma + \sigma^\gamma - \sigma^{\gamma/2}), \\ u_5 &= \sigma + \sigma^\gamma + \sigma^{\gamma/2}, \end{aligned}$$

and study each case separately: the integrals along the intervals $[u_1, u_2]$ and $[u_4, u_5]$, can be estimated by taking absolute values inside; for the other intervals, apply Van der Corput's lemma as usual.

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